On the series that represent tides and surges in an estuary

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(*Received* 13 *August* 1957)

SUMMARY

This paper is concerned with a progressive wave of general form in an infinitely long estuary of uniform cross-section when there is a permanent current independent of the wave. The only approximation is the neglect of friction. Explicit formulae in the form of infinite series are found for the surface-elevation and for the current.

In the special case in which there is no permanent current and when the oscillation at the mouth of the estuary reduces to a single harmonic constituent, the first five harmonic shallow water constituents at any place up the estuary are evaluated.

1. INTRODUCTION

In the making of tide-tables for a port, the basis of the method usually employed is that of a series of harmonic constituents. When the port is on an estuary, a number of these constituents are of the type usually called 'shallow water' constituents. For many years Doodson (1957) has used his method of 'harmonic shallow water corrections' in predicting the times and heights of high and low water. This method implies the existence of a much larger number of shallow water constituents than the number of harmonic units on any predicting machine.

It is the object of the present paper to examine the series in the case of an infinitely long estuary of uniform cross-section when there is a permanent current down the estuary and no friction. This is an ideal problem in the three respects of infinity, uniformity and absence of friction. But, because precise results are obtained, the investigation appears to be of value in spite of the limitations indicated.

As the cross-section of the estuary is uniform, the permanent current is also uniform. Owing to the absence of reflection, the motion takes the form of a progressive wave, and no serious limitation is placed on the shape of this wave at the mouth of the estuary.

The simple basin specified is that which was taken by Saint-Venant (1871), McCowan (1892) and Fjeldstad (1941) when they evolved their accurate finite solutions of the fundamental equations. Saint-Venant did not include a permanent current, but McCowan did. Their solutions show that the heights of high and low water at any place up the estuary are the same as at the mouth, and this gives a complete answer to the most important question regarding a combination of tide and surge. They also show that a bore will always form, the position of its formation depending on the size and shape of the oscillation at the mouth.

But the solutions of Saint-Venant, McCowan and Fjeldstad do not express the surface-elevation as an explicit function of position and time, and it is just such a function which is required in tidal prediction and which is provided by series.

I first obtained the general formulae of the present paper by substituting series with unknown coefficients into the fundamental differential equations. But Dr M. S. Longuet-Higgins then pointed out that the results could be obtained much more shortly from the implicit solutions by means of Lagrange's theorem on implicit functions. I now follow this method.

The early terms of the series have an interest in themselves, and they take simple forms when there is no permanent current. Taking the special case in which the oscillation at the mouth reduces to a single harmonic constituent, the first five harmonic shallow water constituents are evaluated for any place up the estuary. The first three of these were given by Airy in 1842.

2. Notation and general equations

Denote by:

- g the acceleration of gravity;
- h the undisturbed depth of water;
- x distance up the estuary from the mouth;
- t the time;
- ζ the elevation of the water-surface above its undisturbed level;
- u the current up the estuary;
- u_0 the permanent current down the estuary, assumed constant and uniform;
- $F(t)$ the ratio ζ/h at the mouth of the estuary;

and write

$$c = (gh)^{1/2}, \qquad \theta = x/(c-u_0), \qquad \eta = t-\theta.$$

The equation of continuity is

$$\frac{\partial}{\partial x}\{(h+\zeta)u\}+\frac{\partial \zeta}{\partial t} = 0, \tag{1}$$

and the equation of motion is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial \zeta}{\partial x} = 0, \tag{2}$$

while a stationary solution of these equations is given by

$$\zeta = 0, \qquad u = -u_0. \tag{3}$$

Following Lamb (1932, § 187), multiply (1) by $[g/(h+\zeta)]^{1/2}$ and subtract from (2); the resulting equation may be written

$$\frac{\partial u}{\partial t} - \left(\frac{g}{h+\zeta}\right)^{1/2} \frac{\partial \zeta}{\partial t} + \{u - [g(h+\zeta)]^{1/2}\} \left\{\frac{\partial u}{\partial x} - \left(\frac{g}{h+\zeta}\right)^{1/2} \frac{\partial \zeta}{\partial x}\right\} = 0,$$

and this will be satisfied when

$$\frac{\partial u}{\partial t} - \left(\frac{g}{h+\zeta}\right)^{1/2} \frac{\partial \zeta}{\partial t} = 0, \tag{4}$$

$$\frac{\partial u}{\partial x} - \left(\frac{g}{h+\zeta}\right)^{1/2} \frac{\partial \zeta}{\partial x} = 0. \tag{5}$$

Both (4) and (5) may be integrated to give

$$u + u_0 = 2[g(h+\zeta)]^{1/2} - 2c, \tag{6}$$

where the constant of integration has been chosen so as to satisfy (3). Equation (6) may be written in the forms

$$\frac{u+u_0}{c} = 2\left(1+\frac{\zeta}{h}\right)^{1/2} - 2, \tag{7}$$

$$\frac{\zeta}{h} = \frac{u+u_0}{c} + \frac{1}{4}\left(\frac{u+u_0}{c}\right)^2. \tag{8}$$

Substitution from (8) into (2) gives

$$\frac{\partial u}{\partial t} + (c + \tfrac{1}{2}u_0 + \tfrac{3}{2}u)\frac{\partial u}{\partial x} = 0$$

and into (1) gives

$$\left(1 + \frac{u+u_0}{2c}\right)\left\{\frac{\partial u}{\partial t} + (c + \tfrac{1}{2}u_0 + \tfrac{3}{2}u)\frac{\partial u}{\partial x}\right\} = 0,$$

and hence both (1) and (2) reduce, in effect, to

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = 0, \tag{9}$$

where

$$U = c - u_0 + \tfrac{3}{2}(u+u_0). \tag{10}$$

Substitution from (4), (5) for $\partial u/\partial t$, $\partial u/\partial x$ into (9) gives

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} = 0, \tag{11}$$

and from (7) into (10) gives

$$\frac{U}{c} = 3\left(1+\frac{\zeta}{h}\right)^{1/2} - \frac{u_0}{c} - 2. \tag{12}$$

The velocity U is that of propagation of an element of the wave up the estuary.

It is easily verified that solutions of the differential equations (11) and (9) are given respectively by

$$\zeta/h = F(t-x/U), \tag{13}$$

$$(u+u_0)/c = f(t-x/U), \tag{14}$$

where $F(\ )$, $f(\ )$ are any differentiable functions. Since $x = 0$ at the mouth of the estuary, the function F is that which has been defined above. The function $F(t)$ will be regarded as prescribed, and from its physical meaning it follows that $F(t) > -1$. From (7) it follows that

$$\tfrac{1}{2}f = (1+F)^{1/2} - 1, \tag{15}$$

the arguments of F, f being the same.

Equation (13) is equivalent to McCowan's solution (1892), and when $u_0 = 0$ it is equivalent to Saint-Venant's solution (1871).

It is seen from (12) and (13) that, for the same time, the effect of the permanent current is to make a specified surface-elevation occur at a shorter distance up the estuary than when there is no such current.

3. The series

The equations (13) and (14) give ζ and u as implicit functions of t and x, but it is the object of this paper to obtain ζ and u as explicit functions. Such explicit functions, in the form of infinite series, may be derived from (13) and (14) by means of Lagrange's theorem on the expansion of a function which is defined implicitly (Edwards 1896, p. 451).

Lagrange's theorem may be stated as follows. Let $\phi(y)$, $\psi(y)$ be functions which may be expanded in powers of $y-\eta$, and let

$$y = \eta + \theta\,\phi(y);$$

then

$$\psi(y) = \psi(\eta) + \sum_{n=1}^{\infty} \frac{\theta^n}{n!}[\{\phi(\eta)\}^n\,\psi^{(1)}(\eta)]^{(n-1)}, \tag{16}$$

where the bracketed indices denote derivatives with respect to η.

Now, in the present application of the theorem, the parameters η and θ are given the meanings indicated at the beginning of §2, so that

$$t - \frac{x}{U} = \eta + \theta\left(1 - \frac{c-u_0}{U}\right) = y,$$

on taking

$$\phi(y) = 1 - \frac{c-u_0}{U}; \tag{17}$$

thus (13) and (14) may be written as

$$\zeta/h = F(y), \tag{18}$$

$$(u+u_0)/c = f(y). \tag{19}$$

From (12), (17) and (18) it follows that

$$\phi(y) = 1 - \frac{c-u_0}{3c[1+F(y)]^{1/2} - u_0 - 2c}$$
$$= \frac{[1+F(y)]^{1/2} - 1}{[1+F(y)]^{1/2} - \frac{1}{3}u_0/c - \frac{2}{3}}, \tag{20}$$

and from (15) that

$$f^{(1)}(y) = \frac{F^{(1)}(y)}{[1+F(y)]^{1/2}}. \tag{21}$$

It will be supposed that $F(y)$ and $f(y)$ satisfy the condition imposed on $\psi(y)$ in Lagrange's theorem. From (18) and (20), the expansion (16), with F in place of ψ, becomes

$$\frac{\zeta}{h} = F(\eta) + \sum_{n=1}^{\infty} \theta^n Z_n(\eta), \tag{22}$$

where

$$Z_n(\eta) = \frac{1}{n!}\left[\left\{\frac{[1+F(\eta)]^{1/2}-1}{[1+F(\eta)]^{1/2}-\frac{1}{3}u_0/c-\frac{2}{3}}\right\}^n F^{(1)}(\eta)\right]^{(n-1)} \tag{23}$$

for $n = 1, 2, 3, \ldots$, and from (15), (19), (20) and (21), the expansion (16), with f in place of ψ, becomes

$$\frac{u+u_0}{c} = 2[1+F(\eta)]^{1/2} - 2 + \sum_{n=1}^{\infty} \theta^n U_n(\eta), \tag{24}$$

where

$$U_n(\eta) = \frac{1}{n!}\left[\left\{\frac{[1+F(\eta)]^{1/2}-1}{[1+F(\eta)]^{1/2}-\frac{1}{3}u_0/c-\frac{2}{3}}\right\}^n \frac{F^{(1)}(\eta)}{[1+F(\eta)]^{1/2}}\right]^{(n-1)}. \tag{25}$$

The series (22) and (24), with their coefficients given by (23) and (25), constitute the required solution of the fundamental equations in terms of $F(\ )$, which itself gives the prescribed elevation at the mouth of the estuary.

The condition imposed on $\phi(y)$ in Lagrange's theorem requires that $\phi(y)$ shall not be infinite, and, by (17), this requires that U shall not vanish. Assuming that U, the velocity of propagation up the estuary, is once positive, it follows that, for the validity of the expansions (22) and (24), it must always be positive. From (12) and (18) this requires that

$$3[1+F(y)]^{1/2} - u_0/c - 2 > 0. \tag{26}$$

When the expressions (23) and (25) for the coefficients $Z_n(\eta)$ and $U_n(\eta)$ are expanded in ascending powers of $F(\eta)$, the first term in each is

$$\frac{1}{n!}\left[\left\{\frac{3F(\eta)}{2(1-u_0/c)}\right\}^n F^{(1)}(\eta)\right]^{(n-1)},$$

and this is equal to

$$\frac{1}{(n+1)!}\left\{\frac{3}{2(1-u_0/c)}\right\}^n [\{F(\eta)\}^{n+1}]^{(n)}.$$

Examples of this will occur in § 4.

The expansion of $(1+F)^{1/2}$ in ascending powers of F requires that $|F| \leqslant 1$. On combining this with the condition (26) it follows that, for the validity of the expansions,

$$\frac{1}{9}\left(\frac{u_0}{c}+2\right)^2 < 1 + F(\eta) \leqslant 2. \tag{27}$$

At the mouth of the estuary, the relationships (27) give the lower and upper bounds of the ratio of the total disturbed depth of water to the undisturbed depth.

4. EARLY TERMS OF THE SERIES WHEN THERE IS NO PERMANENT CURRENT

When $u_0 = 0$, then

$$\theta = x/c, \qquad \eta = t - x/c,$$

and the early terms of the series (22) and (24) may be written as

$$\frac{\zeta}{h} = F + \frac{x}{c} Z_1 + \frac{x^2}{c^2} Z_2 + \frac{x^3}{c^3} Z_3 + \frac{x^4}{c^4} Z_4, \tag{28}$$

$$\frac{u}{c} = U_0 + \frac{x}{c} U_1 + \frac{x^2}{c^2} U_2 + \frac{x^3}{c^3} U_3 + \frac{x^4}{c^4} U_4, \tag{29}$$

respectively, where $U_0 = 2(1+F)^{1/2} - 2$, and the arguments of F, Z_n, U_n are $t - x/c$.

On using (23) and (25) and expanding Z_n, U_n in ascending powers of F as far as F^5, the following results are obtained:

$$\left.\begin{aligned}
Z_1 &= \left[\frac{3}{4}F^2 - \frac{7}{8}F^3 + \frac{75}{64}F^4 - \frac{1077}{640}F^5\right]^{(1)}, \\
Z_2 &= \left[\frac{3}{8}F^3 - \frac{63}{64}F^4 + \frac{1341}{640}F^5\right]^{(2)}, \\
Z_3 &= \left[\frac{9}{64}F^4 - \frac{189}{320}F^5\right]^{(3)}, \\
Z_4 &= \left[\frac{27}{640}F^5\right]^{(4)}, \\
U_0 &= F - \frac{1}{4}F^2 + \frac{1}{8}F^3 - \frac{5}{64}F^4 + \frac{7}{128}F^5, \\
U_1 &= \left[\frac{3}{4}F^2 - \frac{9}{8}F^3 + \frac{105}{64}F^4 - \frac{1563}{640}F^5\right]^{(1)}, \\
U_2 &= \left[\frac{3}{8}F^3 - \frac{9}{8}F^4 + \frac{1647}{640}F^5\right]^{(2)}, \\
U_3 &= \left[\frac{9}{64}F^4 - \frac{207}{320}F^5\right]^{(3)}, \\
U_4 &= \left[\frac{27}{640}F^5\right]^{(4)}.
\end{aligned}\right\} \tag{30}$$

These results may also be obtained from the fundamental equations by the method of successive approximations.

5. HARMONIC CONSTITUENTS

When there is no surge, the surface-elevation at the mouth of the estuary is given by a series of the form

$$F(t) = \sum_{n=1}^{\infty} A_n \cos(\sigma_n t + \alpha_n), \tag{31}$$

where A_n, σ_n, α_n are constants. When (31) is substituted into the formulae of § 3, the surface-elevation at any place up the estuary is given by a triple series of harmonic constituents.

When the series (31) reduces to one term, so that $F(t) = A\cos(\sigma t + \alpha)$, then $F(t - x/c) = A\cos\theta$, where $\theta = \sigma(t - x/c) + \alpha$. Substitution into the results of §4 then gives

$$\left.\begin{aligned}
\frac{Z_1}{\sigma} &= -\frac{3}{4}A^2\sin 2\theta + \frac{21}{32}A^3(\sin\theta + \sin 3\theta) - \\
&\quad -\frac{75}{128}A^4(2\sin 2\theta + \sin 4\theta) + \frac{1077}{2048}A^5(2\sin\theta + 3\sin 3\theta + \sin 5\theta),\\
\frac{Z_2}{\sigma^2} &= -\frac{9}{32}A^3(\cos\theta + 3\cos 3\theta) + \frac{63}{32}A^4(\cos 2\theta + \cos 4\theta) - \\
&\qquad -\frac{1341}{2048}A^5(2\cos\theta + 9\cos 3\theta + 5\cos 5\theta),\\
\frac{Z_3}{\sigma^3} &= \frac{9}{16}A^4(\sin 2\theta + 2\sin 4\theta) - \frac{189}{1024}A^5(2\sin\theta + 27\sin 3\theta + 25\sin 5\theta),\\
\frac{Z_4}{\sigma^4} &= \frac{27}{2048}A^5(2\cos\theta + 81\cos 3\theta + 125\cos 5\theta).
\end{aligned}\right\} \quad (32)$$

The terms of the above formulae in A^2 and A^3 are equivalent to formulae given by Airy in (1842).

Rearrangement of the series (28) so as to group together terms with the same function of θ gives

$$\begin{aligned}
\frac{\zeta}{h} = A&\left\{1 - \frac{9}{32}A^2\left(\frac{\sigma x}{c}\right)^2\left[1 + \frac{1}{32}A^2\left\{149 - 3\left(\frac{\sigma x}{c}\right)^2\right\}\right]\right\}\cos\theta + \\
&+ \frac{1}{32}A^3\frac{\sigma x}{c}\left\{21 + \frac{1}{32}A^2\left[1077 - 378\left(\frac{\sigma x}{c}\right)^2\right]\right\}\sin\theta + \\
&+ \frac{63}{32}A^4\left(\frac{\sigma x}{c}\right)^2\cos 2\theta - \\
&- \frac{3}{4}A^2\frac{\sigma x}{c}\left\{1 + \frac{1}{4}A^2\left[\frac{25}{4} - 3\left(\frac{\sigma x}{c}\right)^2\right]\right\}\sin 2\theta - \\
&- \frac{27}{32}A^3\left(\frac{\sigma x}{c}\right)^2\left\{1 + \frac{3}{64}A^2\left[149 - 27\left(\frac{\sigma x}{c}\right)^2\right]\right\}\cos 3\theta + \\
&+ \frac{3}{32}A^3\frac{\sigma x}{c}\left\{7 + \frac{3}{32}A^2\left[\frac{359}{2} - 567\left(\frac{\sigma x}{c}\right)^2\right]\right\}\sin 3\theta + \\
&+ \frac{63}{32}A^4\left(\frac{\sigma x}{c}\right)^2\cos 4\theta - \frac{3}{8}A^4\frac{\sigma x}{c}\left\{\frac{25}{16} - 3\left(\frac{\sigma x}{c}\right)^2\right\}\sin 4\theta - \\
&- \frac{45}{2048}A^5\left(\frac{\sigma x}{c}\right)^2\left\{149 - 75\left(\frac{\sigma x}{c}\right)^2\right\}\cos 5\theta + \\
&\qquad + \frac{3}{1024}A^5\frac{\sigma x}{c}\left\{\frac{359}{2} - 1575\left(\frac{\sigma x}{c}\right)^2\right\}\sin 5\theta. \quad (33)
\end{aligned}$$

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